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INDEX

The Study of Exponentiated Gumbel Distribution and Related Inference Through Simulation

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ABSTRACT

Two parameter Exponentiated Gumbel (EG) distribution is a right skewed unimodal distribution. We discuss point and interval estimation of parameters of EG distribution by the method of maximum likelihood and provide an expression for the Fisher information matrix. A bootstrap method to obtain confidence interval is also discussed. Inference for R=P(Y<X) is provided when X and Y are independently but not identically EG distributed random variables. Testing for R based on exact and asymptotic distribution is discussed along with simulation study.

KEYWORDS

Maximum likelihood estimator, Fisher information matrix, uniformly minimum variance unbiased estimator and Bayes' estimator.

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1. INTRODUCTION

In literature, exponentiated family of distribution defined in two ways. If $F(x/\underline{\theta})$ is cumulative distribution function (c.d.f.) of base line distribution then by adding one more parameter (say α), the c.d.f. of exponentiated base line distribution is $G(x/\underline{\theta},\alpha)$ given by

(a) $G(x/\underline{\theta},\alpha)= [\ F(x/\underline{\theta})\ ]^{\alpha}$, $\alpha>0$, $\underline{\theta}\in\Theta$ and $x\in R$.

(b) $G(x/\underline{\theta},\alpha)= 1-[1- F(x/\underline{\theta})\ ]^{\alpha}$, $\alpha>0$, $\underline{\theta}\in\Theta$ and $x\in R$.

Gupta et al. (1998) introduced the Exponentiated Exponential (EE) distribution as a generalization of the standard Exponential distribution. The two parameter EE distribution associated with definition (a) above, have been studied in detail by Gupta and Kundu (2001) which is a sub-model of the Exponentiated Weibull distribution, introduced by Mudholkar and Shrivastava (1993). S. Nadarajah (2006) introduced Exponentiated Gumbel (EG) distribution using (b) above.

The cumulative distribution function of the EG distribution is defined by $F(x;\alpha,\sigma)=(G(x,\sigma))^{\alpha}=\left(\exp\left(-e^{-\frac{x}{\sigma}}\right)\right)^{\alpha}$

$,\alpha, \sigma>0\ , -\infty<x<\infty$ (1.1)

which is simply the α^{th} power of c.d.f. of the Gumbel distribution.

The Probability density function (p.d.f.) corresponding to (1.1) is

$$f(x;\alpha,\sigma)=\frac{\alpha}{\sigma}\left(\exp\left(-e^{-\frac{x}{\sigma}}\right)\right)^{\alpha} e^{-\frac{x}{\sigma}}$$

$, \alpha, \sigma>0\ , -\infty<x<\infty$ (1.2)

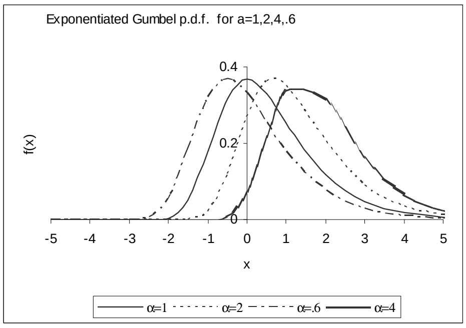

Figure-1. Probability density function.

We shall write x ~ EG(α,σ) to denote an absolutely continuous random variable X having the EG distribution with shape and scale parameters are α and σ respectively whose p.d.f. is given by (1.2). The shapes of p.d.f. for EG distribution with scale parameter σ=1 and various values of parameter α (=1, 2, 4, 0.6) are shown in the above Figures. Fig. 1 shows that it is an unimodal and right skewed density function.

2. MAXIMUM LIKELIHOOD ESTIMATOR AND THE FISHER INFORMATION MATRIX

Suppose $X_1,X_2,\ldots..X_n$ is a random sample from EG(α,σ). Therefore, the log-likelihood function L for the observed sample is

$$L=n\ln\alpha-n\ln\sigma-\frac{1}{\sigma}\sum_{i=1}^{n}x_i-\alpha\sum_{i=1}^{n}e^{-\frac{x_i}{\sigma}} \quad (2.1)$$

Therefore, to obtain the MLE's of α and σ , either we can maximize (2.1) directly with respect to α and σ or we can solve the non-linear normal equations which are

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}} = 0 \qquad (2.2)$$

$$\frac{\partial L}{\partial \sigma} = \frac{-n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - \frac{\alpha}{\sigma^2}\sum_{i=1}^{n} x_i \, e^{-\frac{x_i}{\sigma}} = 0 \qquad (2.3)$$

From (2.2), we obtain the MLE's of α as a function of σ, say $\hat{\alpha}(\sigma)$ as

$$\hat{\alpha}(\sigma) = \frac{n}{\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}} \qquad (2.4)$$

Case 1: If the scale parameter is known (say σ=1), the MLE of the parameter α can be obtained directly from (2.4).

Lemma (2.1): For known scale parameter (say σ=1) the p.d.f. of $\hat{\alpha}$ is

$$f_Y(y,\alpha) = \frac{1}{n! \quad \alpha}\left(\frac{n\alpha}{y}\right)^{n+1} e^{-\frac{n\alpha}{y}} \;, \; y>0 \qquad (2.5)$$

Proof : Suppose $W = \left(-2\alpha \sum \ln(\exp(-e^{-x_i}))\right)$ then W has chi-square distribution with 2n d.f., since $\left(\exp(-e^{-x_i})\right)^{\alpha}$ is c.d.f. of standard EG distribution and follows uniform distribution over (0,1). Let $Y = \frac{2n\alpha}{W}$, then c.d.f. of Y is given as

$$P(Y \le y) = P\left(\frac{2n\alpha}{W} \le y\right) = 1 - P\left(W \le \frac{2n\alpha}{y}\right) \qquad (2.6)$$

Using Chi-square distribution , the p.d.f. corresponding to (2.6) is

$$f_Y(y,\alpha) = \frac{1}{n!\alpha}\left(\frac{n\alpha}{y}\right)^{n+1} e^{-\frac{n\alpha}{y}} \;, y>0$$

Lemma (2.2): For known scale parameter (say σ=1) , the 100(1-δ)% confidence interval of α is given by

$$\left(\frac{Y}{2n}\chi^2_{2n,\delta/2}, \quad \frac{Y}{2n}\chi^2_{2n,1-\delta/2}\right).$$

Case 2: If both the parameters are unknown, first the estimate of the scale parameter can be obtained by using maximum likelihood estimation method

$$L\,(\hat{\alpha}(\sigma),\sigma)= C - n\ln\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}} - n\ln\sigma - \frac{1}{\sigma}\sum_{i=1}^{n} x_i \qquad (2.7)$$

With respect to σ. Here C is a constant independent of σ. Once $\hat{\sigma}$ is obtained, $\hat{\alpha}$ can be obtained from (2.4) as $\hat{\alpha}(\sigma)$. Therefore, it reduces the two-dimensional problem to a one-dimensional problem.

In this situation we use the asymptotic normality result to obtain the asymptotic confidence interval. We can state the result as follows.

$\sqrt{n}(\hat{\theta}-\theta) \rightarrow N_2(0, I^{-1}(\theta))$ where I(θ) is the Fisher Information matrix.

$$I(\theta) = \frac{-1}{n}\begin{bmatrix} E\left(\frac{\partial^2 L}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 L}{\partial\alpha\partial\sigma}\right) \\ E\left(\frac{\partial^2 L}{\partial\sigma\partial\alpha}\right) & E\left(\frac{\partial^2 L}{\partial\sigma^2}\right) \end{bmatrix}$$ and $\hat{\theta} = (\hat{\alpha}, \hat{\sigma})$, θ=(α,σ),

$$E\left(\frac{\partial^2 L}{\partial\alpha^2}\right) = \frac{-n}{\alpha^2}, \qquad E\left(\frac{\partial^2 L}{\partial\alpha\partial\sigma}\right) = \frac{\alpha}{\sigma 2^{\alpha}}\sum_{i=1}^{n} E(\ln u_i),$$

$$E\left(\frac{\partial^2 L}{\partial\sigma^2}\right) = \frac{n}{\sigma^2} - \frac{2}{\sigma^2}\sum_{i=1}^{n} E(\ln v_i) - \frac{2\alpha^2}{2^{\alpha}\sigma^2}\sum_{i=1}^{n} E(\ln u_i) - \frac{\alpha^2}{2^{\alpha}\sigma^2}\sum_{i=1}^{n} E(\ln u_i)^2$$

where u_i and v_i has gamma distribution with parameters (2,α) and (1,α) respectively. Since θ is unknown, $I^{-1}(\theta)$ is estimated by replacing θ with its MLE and this can be used to obtain the asymptotic confidence intervals of α and σ.

2.1. Bootstrap Confidence Interval:

In this subsection, we propose a percentile bootstrap method (Efron, 1982) for constructing confidence interval of α and σ which is as follows.

Step-1: Generate random samples $x_1, x_2, \ldots, x_n$ from EG(α,σ) and compute $\hat{\alpha}$ and $\hat{\sigma}$ using maximum likelihood method.

Step-2: Using $\hat{\alpha}$ and $\hat{\sigma}$ generate a bootstrap sample $x_1^*, x_2^*, \ldots, x_n^*$ from ES($\hat{\alpha}$, $\hat{\sigma}$). Based on bootstrap samples compute bootstrap estimate $\hat{\alpha}^*$ and $\hat{\sigma}^*$.

Step-3: Repeat step-2 NBOOT times (usually NBOOT=1000).

Step-4: Compute cumulative distribution function of $\hat{\alpha}^*$ and $\hat{\sigma}^*$, say H(x) and G(x) respectively , where H(x)= P($\hat{\alpha}^* \leq$ x) and $\hat{\alpha}_{Boot-p}(x) = H^{-1}(x)$ and G(x)= P($\hat{\sigma}^* \leq$ x) and $\hat{\sigma}_{Boot-p}(x) = G^{-1}(x)$ for a given x. The approximate 100(1-δ)% bootstrap confidence intervals for α and σ are given by

$\left(\hat{\alpha}_{Boot-p}(\delta/2)\,,\, \hat{\alpha}_{Boot-p}(1-\delta/2)\right)$ and $\left(\hat{\sigma}_{Boot-p}(\delta/2)\,,\, \hat{\sigma}_{Boot-p}(1-\delta/2)\right)$ respectively.

3. POINT AND INTERVAL ESTIMATION OF R

Now we consider the problem of estimating R=P(Y<X) when X and Y are independent EG random variables with shape, scale parameters α , σ and β , σ respectively then R=P(Y<X) $= \dfrac{\alpha}{\alpha+\beta}$

Case 1: When scale parameter σ is unknown.

Suppose $X_1, X_2, \ldots X_n$ is a random sample from EG(α,σ) and $Y_1, Y_2, \ldots Y_m$ is a random sample from EG(β,σ). Therefore, the log-likelihood function L of α, β and σ for the observed sample is

$$L = n\ln\alpha - \alpha\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}} + m\ln\beta - \beta\sum_{j=1}^{m} e^{-\frac{y_j}{\sigma}} - (m+n)\ln\sigma - \frac{\left(\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j\right)}{\sigma}, \qquad (3.1)$$

hence MLE's of α and β as $\hat{\alpha} = \dfrac{n}{\sum_{i=1}^{n}\exp(-\frac{x_i}{\hat{\sigma}})}$ and $\hat{\beta} = \dfrac{m}{\sum_{j=1}^{m}\exp(-\frac{y_j}{\hat{\sigma}})}$.

Therefore, the MLE of R namely $\hat{R}_1$ is given by

$$\hat{R}_1 = \frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}} \qquad (3.2)$$

Now to obtain asymptotic distribution of R, we first obtain the asymptotic distribution of $(\hat{\alpha}, \hat{\beta}, \hat{\sigma})$. Based on the asymptotic distribution of $\hat{R}$, we obtain asymptotic confidence interval of R. Let us denote the Fisher Information matrix of (α,β,σ) as I(α,β,σ) where

$$I(\alpha,\beta,\sigma) = -\begin{bmatrix} E\left(\frac{\partial^2 L}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 L}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 L}{\partial\alpha\partial\sigma}\right) \\ E\left(\frac{\partial^2 L}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 L}{\partial\beta^2}\right) & E\left(\frac{\partial^2 L}{\partial\beta\partial\sigma}\right) \\ E\left(\frac{\partial^2 L}{\partial\sigma\partial\alpha}\right) & E\left(\frac{\partial^2 L}{\partial\sigma\partial\beta}\right) & E\left(\frac{\partial^2 L}{\partial\sigma^2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \quad \text{say.}$$

Moreover $E\left(\frac{\partial^2 L}{\partial\alpha^2}\right) = -\frac{n}{\alpha^2}$ and $E\left(\frac{\partial^2 L}{\partial\beta^2}\right) = -\frac{m}{\beta^2}$, $E\left(\frac{\partial^2 L}{\partial\alpha\partial\beta}\right) = E\left(\frac{\partial^2 L}{\partial\beta\partial\alpha}\right) = 0$,

$$E\left(\frac{\partial^2 L}{\partial\alpha\partial\sigma}\right) = \frac{\alpha}{\sigma\, 2^{\alpha}}\sum_{i=1}^{n} E(\ln u_i) = E\left(\frac{\partial^2 L}{\partial\sigma\partial\alpha}\right) \quad E\left(\frac{\partial^2 L}{\partial\beta\partial\sigma}\right) = \frac{\beta}{\sigma\, 2^{\beta}}\sum_{j=1}^{m} E(\ln v_j) = E\left(\frac{\partial^2 L}{\partial\sigma\partial\alpha}\right)$$

$$E\left(\frac{\partial^2 L}{\partial\sigma^2}\right) = \frac{n}{\sigma^2} - \frac{2}{\sigma^2}\sum_{i=1}^{n} E(\ln w_i) - \frac{\alpha^2}{\sigma^2\, 2^{\alpha-1}}\sum_{i=1}^{n} E(\ln u_i) - \frac{\alpha^2}{\sigma^2\, 2^{\alpha}}\sum_{i=1}^{n} E(\ln u_i)^2$$

$$+\frac{m}{\sigma^2} - \frac{2}{\sigma^2}\sum_{j=1}^{m} E(\ln z_j) - \frac{\beta^2}{\sigma^2\, 2^{\beta-1}}\sum_{j=1}^{m} E(\ln v_j) - \frac{\beta^2}{\sigma^2\, 2^{\beta}}\sum_{j=1}^{m} E(\ln v_j)^2$$

where u_i and v_j has gamma $(2,\alpha)$ and $(2,\beta)$ and w_i and z_j has exponential α and β distribution respectively.

Theorem 1: As m, n $\to\infty$ and $\frac{m}{n} \to p$ then

$$\left((\hat{\alpha}-\alpha), (\hat{\beta}-\beta), (\hat{\sigma}-\sigma)\right) \to N_3(0, A(\alpha,\beta,\sigma)),$$

where $A(\alpha,\beta,\theta) = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and elements of A(α ,β, σ) are the corresponding elements of the inverse of the Fisher Information matrix I(α, β, σ).

Proof : Proof follows from asymptotic properties of MLEs under regularity conditions and multivariate central limit theorem.

Theorem 2: As m, n $\to\infty$ and $\frac{n}{m} \to p$ then $\sqrt{n}(\hat{R}-R) \to N(0,B)$, where

$$B=\frac{1}{u(\alpha+\beta)^4}\left(\beta^2\left(a_{22}a_{33}-a_{23}^2\right)-2\alpha\beta\sqrt{p}a_{23}a_{31}+\alpha^2 p\left(a_{11}a_{33}-a_{13}^2\right)\right)$$

and $u=a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{13}a_{22}a_{31}.$

Proof : Proof follows from invariance property of CAN estimator under continuous transformation, and omitted for brevity.

Using Theorem 2, we can obtain asymptotic confidence interval of R as

$$\left(\hat{R}-Z_{1-\delta/2}\frac{\sqrt{\hat{B}}}{\sqrt{n}}, \quad \hat{R}+Z_{1-\delta/2}\frac{\sqrt{\hat{B}}}{\sqrt{n}}\right) \qquad (3.3)$$

Remark (3.1): To estimate variance B, the empirical Fisher's information matrix and MLEs of α, β and σ may be used. However simulation study due to Kundu and Gupta (2005) for EE distribution indicates that confidence interval defined in (3.3) has comparatively low coverage probability. They have suggested bootstrap method to get a better confidence interval with respect to coverage probability.

Bootstrap confidence interval:

Step-1: Generate random samples $x_1,x_2,\ldots..x_n$ from ES(α,σ) and $y_1,y_2,\ldots.y_m$ from ES(β,σ) and compute $\hat{\alpha},\hat{\beta}$ and $\hat{\sigma}$ using maximum likelihood method.

Step-2: Using $\hat{\alpha}$ and $\hat{\sigma}$ generate a bootstrap sample $x_1^*,x_2^*,\ldots.,x_n^*$ from ES($\hat{\alpha}$, $\hat{\sigma}$) and similarly using $\hat{\beta}$ and $\hat{\sigma}$ generate a bootstrap sample $y_1^*,y_2^*,\ldots.,y_m^*$ from ES($\hat{\beta}$, $\hat{\sigma}$). Based on these bootstrap samples compute bootstrap estimate of R,

$\hat{R}^*=\dfrac{\hat{\alpha}^*}{\hat{\alpha}^*+\hat{\beta}^*}$, where $\hat{\alpha}^*$ and $\hat{\beta}^*$ are the MLEs of α and β obtained from the corresponding bootstrap samples.

Step-3: Repeat step-2 NBOOT times (usually NBOOT=1000).

Step-4: Compute cumulative distribution function of $\hat{R}^*$, say H(x), where

H(x)= P($\hat{R}^*$ ≤ x) and $\hat{R}_{Boot-p}(x)=H^{-1}(x)$ for a given x. The approximate 100(1-δ)% bootstrap confidence interval is given by

$$\left(\hat{R}_{Boot-p}(\delta/2)\,,\,\hat{R}_{Boot-p}(1-\delta/2)\right) \qquad (3.4)$$

Case 2: When scale parameter σ is known.

Without loss of generality, we can assume that σ=1. Suppose $X_1, X_2, \ldots X_n$ is a random sample from EG(α,1) and $Y_1, Y_2, \ldots Y_m$ is a random sample from EG(β,1) and based on the samples we want to estimate R. Based on the above sample, it is clear that , the MLE of R namely $\hat{R}_2$ is given by $\hat{R}_2 = \frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}$ where

$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \exp(-x_i)}$ and $\hat{\beta} = \frac{m}{\sum_{j=1}^{m} \exp(-y_j)}$

Lemma (3.1) : The p.d.f. of $\hat{R}_2$ is given by

$$f_{\hat{R}_2}(r) = \frac{\Gamma(m+n)}{\Gamma m \, \Gamma n} \left(\frac{n}{m}\right)^n \left(\frac{\alpha}{\beta}\right)^{n-1} \frac{\left(\frac{1-r}{r}\right)^{n-1}}{\left(1+\frac{n\alpha}{m\beta}\left(\frac{1-r}{r}\right)\right)^{m+n}},$$

0<r<1 (3.5)

Proof : $\hat{R}_2$ can be expressed as $\hat{R}_2 = \frac{1}{1+\frac{mW}{nV}}$ Where $W = -\sum \ln\left(\exp(-e^{-x_i})\right)$ and $V = -\sum \ln\left(\exp(-e^{-y_j})\right)$. We see that -2αW and -2βV are two independent chi-square random variables with 2n and 2m degrees of freedom (d.f.) respectively. Therefore $\hat{R}_2$ can be rewritten as $\hat{R}_2 = \left(1+\frac{\beta}{\alpha}Z\right)^{-1}$, where Z $= \frac{-2\alpha W/2n}{-2\beta V/2m}$ has F distribution with (2n, 2m) degrees of freedom (d.f.). Therefore p.d.f. of $\hat{R}_2$ can be obtained easily and is as given in equation (3.7).

Lemma (3.2) : An exact 100(1-γ)% confidence interval of R is

$$\left(\left(1+F_{2m,2n;(1-\gamma/2)}\left(\frac{1}{\hat{R}_2}-1\right)\right)^{-1}, \left(1+F_{2m,2n;(\gamma/2)}\left(\frac{1}{\hat{R}_2}-1\right)\right)^{-1}\right) \quad (3.6)$$

Lemma (3.3) : The asymptotic 100(1-γ)% confidence interval of R is

$$\left(\left(\hat{R}_2 - Z_{1-\gamma/2}\sqrt{\frac{m+n}{mn}} \quad \hat{R}_2(1-\hat{R}_2)\right), \left(\hat{R}_2 + Z_{1-\gamma/2}\sqrt{\frac{m+n}{mn}} \quad \hat{R}_2(1-\hat{R}_2)\right)\right) \quad (3.7)$$

where $Z_{1-\gamma/2}$ is the $(1-\gamma/2)^{th}$ quantile of the standard normal distribution.

Proof : The MLE $\hat{R}_2$ is asymptotically normal with mean R and variance

$\sigma^2_{\hat{R}_1} = \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial R}{\partial\theta_i}\frac{\partial R}{\partial\theta_j}I^{-1}_{ij}$ where $(\theta_1,\theta_2) = (\alpha,\beta)$ and I^{-1}_{ij} is the (i,j)[th] element of the inverse of the Fisher's information matrix $I(\alpha,\beta)$ about the parameters (α,β) and

$I(\alpha,\beta) = -\begin{bmatrix} \frac{n}{\alpha^2} & 0 \\ 0 & \frac{m}{\beta^2} \end{bmatrix}$, (See Rao (1965)). It can be seen that, $\sigma^2_{\hat{R}_1} = \left(\frac{m+n}{mn}\right)R^2(1-R)^2$.

Therefore the asymptotic 100(1-γ)% confidence interval of R can be obtained using standardized statistic as a pivotal quantity. We replace 'R' in the asymptotic variance by its MLE.

We perform some simulation experiments using percentile bootstrap method when scale parameter σ is unknown to observe the behavior of the MLE and confidence intervals for various sample sizes and for various values of (α, β). We consider the sample sizes (n, m)= (10,10), (10, 20), (20, 20), (20, 40), (40, 40) and the parameter values α= 2, σ=4 and β = 2, 3, 6 and 8. Average biases and mean squared errors (MSEs) of R are reported over 1000 replications for 1000 bootstrap samples. We compute 95% confidence intervals using (3.4) and estimate coverage percentages and average lengths of confidence interval. The results are reported in Table 1.

We also perform some simulation experiments when scale parameter σ is known (σ=1). We consider the sample sizes (n, m)= (10,10), (10, 20), (20, 20), (20, 40), (40, 40) and the parameter values α= 2 and β = 2, 3, 6 and 8. Average biases and mean squared errors (MSEs) of R are reported over 10000 replications. We compute 95% confidence intervals and estimate coverage percentages and average lengths of both asymptotic and exact confidence interval. The results are reported in Table 2.

Table-1. Biases, MSEs, Confidence Lengths and Coverage Percentages of C. I.

Sample size	**2**	**3**	**6**	**8**
(10, 10)	-0.0058(0.0131) 0.4273(93.00)	-0.0005 (0.0124) 0.4139 (93.00)	-0.0096 (0.0077) 0.3286 (90.70)	-0.0054 (0.0061) 0.2899 (91.40)
(10, 20)	0.0125 (0.0109) 0.3748 (92.40)	0.0095 (0.0097) 0.3672 (0.9410)	0.0088 (0.0067) 0.3052 (93.10)	0.0011 (0.0050) 0.2643 (92.90)

(20, 20)	-0.0018 (0.0067) 0.3120 (93.70)	-0.0018 (0.0070) 0.3013 (92.80)	-0.0044 (0.0046) 0.2454 (91.50)	-0.0062 (0.0031) 0.2144 (92.30)
(20, 40)	0.0057 (0.0050) 0.2706 (94.00)	0.0067 (0.0050) 0.2630 (93.50)	0.0031 (0.0033) 0.2175 (93.90)	-0.0001 (0.0026) 0.1909 (93.40)
(40, 40)	0.0012 (0.0033) 0.2205 (94.20)	-0.0032 (0.0031) 0.2134 (94.40)	-0.0049 (0.0021) 0.1762 (93.90)	-0.0028 (0.0016) 0.1567 (93.60)

(The first row represent the average biases and MSEs. Second row represent the average length, coverage percentages of the corresponding asymptotic bootstrap confidence interval.)

Table-2. Biases, MSEs, Confidence Lengths and Coverage Percentages of C. I.

Sample size	**2**	**3**	**6**	**8**
(10, 10)	-0.0003(0.0119) 0.4174(91.47) 0.4058(94.83)	0.0033(0.0110) 0.4027(91.86) 0.3935(95.22)	0.0087(0.0073) 0.3237(91.77) 0.3258(95.30)	0.0098(0.0056) 0.2810(91.50) 0.2876(94.93)
(10, 20)	0.0042(0.0090) 0.3659(92.60) 0.3581(94.70)	0.0093(0.0086) 0.3542(92.30) 0.3507(94.46)	0.0120(0.0057) 0.2851(93.52) 0.2927(94.72)	0.0105(0.0043) 0.2459(92.90) 0.2572(94.74)
(20, 20)	-0.0018(0.0060) 0.3024(93.45) 0.2977(95.02)	0.0016(0.0057) 0.2909(93.11) 0.2872(94.78)	0.0056(0.0037) 0.2313(93.15) 0.2323(94.61)	0.0045(0.0026) 0.1984(93.49) 0.2012(95.18)
(20, 40)	0.0025(0.0045) 0.2636(94.03) 0.2605(95.15)	0.0040(0.0043) 0.2539(93.83) 0.2527(94.98)	0.006290.0027) 0.2017994.220 0.2048(94.910	0.0056(0.0021) 0.1732(94.10) 0.1776(94.92)
(40, 40)	-	0.0016(0.0028)	0.0018(0.0018)	0.0021(0.0013)

	0.0009(0.0030) 0.2165(94.57) 0.2147(95.39)	0.2082(94.25) 0.2068(95.15)	0.1636(94.22) 0.1640(95.04)	0.1402(94.40) 0.1413(95.23)

(The first rows represent the average biases and the corresponding MSEs are reported within brackets. Second and third rows represent the average lengths and the corresponding coverage percentages of the asymptotic and exact confidence intervals respectively.)

Based on the proposed Bootstrap and exact method, the overall findings in Tables 1 and 2 are satisfactory. When sample sizes are increased, bias and MSE decrease for each parameter value, demonstrating the consistency of the method. In each case's coverage probability closely

approximates the confidence coefficient, and the average length of the confidence interval is small and finite.

4. TESTING OF HYPOTHESIS

The EG distribution is ordered with respect to the 'likelihood ratio' ordering ($X \leq_{lr} Y$). Since α and β both are unknown, it will be of interest to know whether $\alpha<\beta$ or not. We put this as a problem of hypothesis testing. We consider test for hypothesis $H_0:\alpha \leq \beta$ against $H_1 : \alpha > \beta$. Equivalently we can test $H_0: R \leq 0.5$ against $H_1: R > 0.5$. Using Lemma (3.3), an asymptotic test of size γ rejects the null hypothesis if, $\left(\hat{R}_2 - \frac{1}{2}\right) > \sqrt{\frac{m+n}{16\,mn}} \quad Z_{1-\gamma}$ (4.1)

where $Z_{1-\gamma}$ is the $(1-\gamma)^{th}$ quantile of the standard normal distribution. Also an exact test of size γ for the above problem, using lemma (3.2), rejects the null hypothesis if $\left(\frac{\hat{R}_2}{1-\hat{R}_2}\right) > F_{2n,2m;1-\gamma}$, (4.2) where $F_{2n,\,2m;\,1-\gamma}$ is the $(1-\gamma)^{th}$ quantile of F distribution with (2n, 2m) d.f. As an independent interest, we can also obtain an asymptotic and exact test of the desired size for alternatives $H'_1: R<0.5$ and $H''_1: R \neq 0.5$.

Through simulation study, comparison of power has been made for two test given in (5.1) and (5.2). The power was determined by generating 1000 random samples of sizes (n, m)=(10,10), (10,20), (20,20), (20,40) and (40,40). The results for the tests at the significance level γ=0.01 and 0.05 are presented in Table 3 and Table 4 respectively. P_1 and P_2 are referred to as power based on asymptotic and exact test as defined in (5.1) and (5.2) respectively.

Table 3 : Power of the test based on asymptotic and exact distribution of R, γ=0.01.

R	**(10, 10)**		**(10, 20)**		**(20, 20)**		**(20, 40)**		**(40, 40)**	
	P_1	P_2	P_1	P_2	P_1	P_2	P_1	P_2	P_1	P_2
0.500	0.006	0.089	0.010	0.020	0.007	0.009	0.012	0.017	0.009	0.010
0.526	0.011	0.016	0.018	0.034	0.019	0.022	0.030	0.040	0.027	0.030
0.555	0.021	0.029	0.039	0.066	0.048	0.042	0.063	0.085	0.082	0.087
0.588	0.039	0.057	0.072	0.113	0.098	0.109	0.142	0.181	0.210	0.223
0.625	0.079	0.105	0.137	0.203	0.208	0.227	0.307	0.368	0.464	0.478
0.666	0.159	0.208	0.257	0.347	0.408	0.434	0.556	0.620	0.766	0.777
0.714	0.301	0.366	0.460	0.567	0.675	0.701	0.839	0.876	0.955	0.958
0.769	0.539	0.606	0.744	0.827	0.914	0.924	0.980	0.987	0.998	0.999
0.833	0.840	0.879	0.956	0.978	0.995	0.996	0.999	0.999	1	1
0.909	0.992	0.995	0.999	0.999	1	1	1	1	1	1

Table 4 : Power of the test based on asymptotic and exact distribution of R, γ=0.05.

R	**(10, 10)**		**(10, 20)**		**(20, 20)**		**(20, 40)**		**(40, 40)**	
	P_1	P_2	P_1	P_2	P_1	P_2	P_1	P_2	P_1	P_2
0.500	0.046	0.050	0.057	0.073	0.046	0.047	0.057	0.065	0.047	0.047
0.526	0.069	0.073	0.088	0.109	0.085	0.087	0.114	0.129	0.116	0.116
0.555	0.119	0.125	0.148	0.179	0.171	0.175	0.207	0.229	0.257	0.258
0.588	0.178	0.189	0.240	0.278	0.293	0.298	0.370	0.402	0.479	0.482
0.625	0.282	0.293	0.376	0.422	0.479	0.485	0.591	0.622	0.728	0.730
0.666	0.431	0.444	0.564	0.610	0.697	0.702	0.815	0.837	0.920	0.921
0.714	0.627	0.639	0.765	0.804	0.881	0.884	0.959	0.967	0.992	0.992
0.769	0.830	0.840	0.931	0.945	0.981	0.982	0.997	0.998	1	1
0.833	0.966	0.968	0.995	0.996	0.999	0.999	1	1	1	1
0.909	0.999	0.999	1	1	1	1	1	1	1	1

It is observed from the simulation study that (i) both the tests perform well with respect to the power. (ii) Power of the test based on exact test is slightly higher than that of asymptotic test. (iii) Both the tests are consistent in the sense that as sample sizes increase, their power show improvement. (iv) A comparison with the usual nonparametric Wilcoxon Mann Whitney test for H_0: $P(Y<X)=0.5$ was made. It is found that parametric procedure (i.e., exact and asymptotic test) have better power than the more general WMW-test.

5. CONCLUSIONS

In this paper we estimate reliability R for Exponentiated Gumbel distribution with different shape parameters and same scale parameter. The performance of the MLE is quite satisfactory in terms of biases and MSEs. It is observed that when sample sizes increase the MSEs decreases. It verifies the consistency property of the MLE of R. The exact distribution of MLE of R is obtained and used for constructing confidence interval. The asymptotic confidence interval based on the MLE of R also works well for samples of sizes greater than or equal to 20. The exact as well as asymptotic test for testing reliability R has been given. The performances of both the tests are satisfactory with respect to the power than usual nonparametric Wilcoxon Mann Whitney test.

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