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# INDEX

# A Review on Nuisance Parameter Free Inferential Procedures for Shape-Scale and Location-Scale Family of Distributions

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***ABSTRACT***

*A variety of parametric and non-parametric inferential procedures are available to study inference on the parameter of interest in the presence of nuisance parameters, but majority of these are constrained by certain limitations, as for example depicted through a variety of examples by Berger (1999). Also, small deviations from the underlying assumptions might often cause biased statistical inference, especially in small to moderate size samples. Additionally, existence of the nuisance parameters also disturbs the statistical properties of the estimation procedures of the parameter of interest. This motivates us to take brief review on improved or efficient and unified superior nuisance parameter-free (invariant) inferential procedures under shape-scale and location-scale family of distributions.*

***KEYWORDS***

*Generalized variable approach, Maximal scale invariant Estimator, Integrated likelihood, Profile likelihood.*

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## 1. INTRODUCTION

Lifetime data are often well modelled by distributions belonging to shape-scale and location-scale families of distributions and are widely used in almost every discipline**,** see for example Kulkarni and Powar (2010, 2011), Patil and Kulkarni (2011), Jones (2015), Powar and Kulkarni (2015), Sengupta et. al. (2015), Rigby et. al. (2005, 2019) and Maswadah (2013, 2022).[1-3] The characteristics of a dataset can be measured through the measures of central tendency, dispersion, skewness, and kurtosis, which are usually well-defined functions of the shape, scale, and location parameters. In this context, we review some efficient or improved inferential procedures for shape-scale and location-scale families.[4] The widely applicable shape-scale families for monitoring lifetime data include the important skewed distributions like Gamma distribution, Weibull distribution, Generalized exponential distribution, Pareto distribution, Log-

logistic, Log-normal distribution, Hyperbolic distribution, Exponentiated exponential, among others. The shape scale family of distributions is characterized by the probability density function (PDF) of the form:

$$g_1\left(x\middle|(a,b)\right) = \frac{1}{a} f_1\left(\frac{x}{a}, b\right), \quad a, b, x > 0.$$

where $a$ and $b$ are the scale and shape parameters respectively, $f_1(.,b)$ being a function of only one parameter, namely the shape parameter $b$.

Distributions belonging to the location-scale family are used in hydrology, bio-statistics, various industrial and analytical fields, among others[5]. Normal, Logistic, Laplace, shifted exponential, Extreme value distribution are some popular members of the location-scale family, among others[6].

The PDF of a random variable Y from a location-scale family of distributions is characterized by density function of the form:

$$g_2\left(\frac{y-\mu}{\sigma}\right) = \frac{1}{\sigma} f_2\left(\frac{y-\mu}{\sigma}\right), \quad y, \mu \,\epsilon\, \mathcal{R}, \sigma > 0.$$

where μ and σ are the location and scale parameters respectively, and $f_2(z)$ is the probability density function of the standard random variable $Z$ having location parameter zero and scale parameter one[7-8].

This article aims to review improved inferential procedures, including point estimation, interval estimation, and hypothesis testing, related to distributions belonging to the location-scale and shape-scale families. Improved inference in the case of point estimation is often related to the reduction of bias and variability of the concerned estimator, while for the case of interval estimation and testing of the hypotheses it concerns the attainment of nominal level, increased coverage probability, and elevated powers, respectively[9-11].

Though often nuisance parameters are absolutely essential for better modeling of the data, most often, existence of one or more nuisance parameters adversely impacts the performance of inference procedures for the parameters of interest. Existence of nuisance parameters may produce their adverse impact in a variety of ways, e.g., increased standard errors of point estimators, volumes/ lengths/ area of confidence region/intervals or rate of convergence of asymptotic properties of the parameters of interest among others[10]. A way-out is an attempt for reducing their impact using some well-known likelihood-based techniques, including conditional likelihood, integrated, profile or pseudo-likelihood function, and their modifications, or through the use of pivot or generalized pivot quantities with completely known probability distributions or circumventing the existence of nuisance parameters through the tricky use of invariance principle[11].

Marginal and conditional likelihoods handle the problem by ignoring some of the data (marginalization) or by ignoring their variability (conditioning). When the number of nuisance parameters are large, then marginalization and conditioning are pretty complex, and sacrifice a sizeable information[12].

In this article, emphasis relies on the procedures eliminating of the impact of nuisance parameters through the invariance principle and generalized variable approach, which are expected to result in more efficient inference procedures by use of the entire data without losing any details [13].

The invariance principle is used to circumvent the effect of the nuisance parameters, making use of their property of being invariant under a group of transformations. The maximal scale invariant inference under a shape-scale family developed by Kulkarni and Patil (2018) turned out to be much efficient than classical procedures for the commonly encountered distributions enjoying the scale invariance property [14]. The generalized variable approach is another efficient tool for exact nuisance-parameters-free parametric inference in certain parametric families. The generalized variable approach is based on the generalized extreme region of a test, the generalization of a data-based extreme region of a test, which depends on the observed data and may involve all the parameters, where the associated p-value is independent of the nuisance parameters [15-16].

In this article, the improved inferences for the inferential problems including point estimation, one sample test and interval estimation for the parameter of interest under the shape-scale family of distributions, stress- strength reliability estimation for the exponentiated-scale family of distributions, test for two-sample comparison for two independent mixed continuous location- scale or some non-location-scale populations and test for homogeneity of variances among several location-scale populations are reviewed[17-19].

In more general set-up, some basic definitions in the generalized pivotal approach are given in the following subsection.

## 2. PRELIMINARIES

### 2.1. The Generalized Variable Approach

Tsui and Weerahandi (1989) introduced the concept of generalized p-values which is based on the generalized pivot quantity (GPQ) and generalized test variable (GTV)[20]. Let $\boldsymbol{X}$ be a random variable with cumulative distribution function (CDF) $F_{\xi}(.)$, where $\boldsymbol{\xi} = (\boldsymbol{\theta}, \boldsymbol{\delta})$ is an unknown parameter vector and $F_{\xi}(.)$ is a member of the shape-scale or location-scale family of distributions. Suppose the interest lies in the parameter $\boldsymbol{\theta}$ while $\boldsymbol{\delta}$ is the nuisance parameter. A GPQ for $\boldsymbol{\theta}$, GTV and generalized p-value (GPV) for testing a one-sided hypothesis $H_0: \boldsymbol{\theta} \leq \boldsymbol{\theta}_0$ verses $H_1: \boldsymbol{\theta} > \boldsymbol{\theta}_0$ is defined below:

***Definition 1: Generalized pivot quantity (GPQ)***

The GPQ $G_{\boldsymbol{\theta}} = \psi(\boldsymbol{X};\ \boldsymbol{x}, \boldsymbol{\xi})$ for $\boldsymbol{\theta}$ is a random quantity that satisfies following two conditions:

i. The distribution of $G_{\boldsymbol{\theta}}$ for given $\boldsymbol{X} = \boldsymbol{x}$ is free from any unknown parameters.

ii. The value of $G_{\boldsymbol{\theta}} = \psi(\boldsymbol{X};\ \boldsymbol{x}, \boldsymbol{\xi})$ at $\boldsymbol{X} = \boldsymbol{x}$ does not depend on any unknown parameter, other than $\boldsymbol{\theta}$. For most of the cases, $G_{\boldsymbol{\theta}} = \boldsymbol{\theta}$ at $\boldsymbol{X} = \boldsymbol{x}$.

The following invariance property of GPQs is an easy consequence of its definition:

***Preposition 1: Invariance property of GPQ***

If $G_{\boldsymbol{\theta}}$ is a GPQ for $\boldsymbol{\theta}$, then for any function $\pi$, $\pi(G_{\boldsymbol{\theta}})$ is GPQ for $\pi(\boldsymbol{\theta})$.

***Definition 2: Generalized test variable (GTV)***

A random quantity $\tau_{\boldsymbol{\theta}} = T\ (\boldsymbol{X};\ \boldsymbol{x}, \boldsymbol{\xi})$ is said to be GTV for the parameter of interest $\boldsymbol{\theta}$ if it satisfies following three properties:

i. The probability distribution of $\tau_{\boldsymbol{\theta}}$ is free from any unknown parameters.

ii. The value of $\tau_{\boldsymbol{\theta}} = T\ (\boldsymbol{X};\ \boldsymbol{x}, \boldsymbol{\xi})$ at $\mathbf{X} = \mathbf{x}$ does not depend on any unknown parameter, other than $\boldsymbol{\theta}$.

iii. For fixed $\mathbf{x}$, the probability $P\ (T\ (\boldsymbol{X};\ \boldsymbol{x}, \boldsymbol{\xi}) \geq t|\boldsymbol{\theta})$, for all $t \geq 0$ is non-decreasing in $\boldsymbol{\theta}$.

***Preposition 2 : Connection between GPQ and GTV***

If $G_{\boldsymbol{\theta}}$ is a GPQ for $\boldsymbol{\theta}$, then $\tau_{\boldsymbol{\theta}} = G_{\boldsymbol{\theta}} - \boldsymbol{\theta}$ is a GTV for $\boldsymbol{\theta}$ (Weerahandi (1995)).

***Definition 3 : Generalized p-value (GPV)***

Based on the GTV defined in Definition 2 and Preposition 2, the generalized

p-value for testing $H_0$ mentioned above is defined by

$p = Sup_{\boldsymbol{\theta} \in H_0} P(T\ (\boldsymbol{X};\ \boldsymbol{x}, \boldsymbol{\theta}, \delta) \geq t)$, were, $t = T\ (\boldsymbol{x};\ \boldsymbol{x}, \boldsymbol{\theta}, \delta)$

$p = P(T\ (\boldsymbol{X};\ \boldsymbol{x}, \boldsymbol{\theta_0}, \delta) \geq t)$,on account of property iii of *Definition 2*.

**2.2. The Invariance Principle**

If $\mathbf{X}$ is a random variable having density function $f(\boldsymbol{x}, \boldsymbol{\theta})$, $\boldsymbol{\theta} \epsilon \boldsymbol{\Theta}$ and $G$ be a group of transformation on the space of values of $\mathbf{X}$ then:

i. $\phi$ is invariant under $G$ if $\phi\big(g(\boldsymbol{x})\big) = \phi(\boldsymbol{x})$ for all $\boldsymbol{x}$ and all $g \epsilon G$.

ii. $T(\boldsymbol{x})$ is maximal invariant under $G$ if $T(\boldsymbol{x_1}) = T(\boldsymbol{x_2}) \Longrightarrow \boldsymbol{x_1} = g(\boldsymbol{x_2})$ for some $g \epsilon G$.

Where $\boldsymbol{x}$ is observed value of **X.**

### *2.2.1 Location Invariant*

Let $x = (x_1, x_2, \dots, x_n)$, be the random sample from location family with location parameter $\mu$ and $G$ be the group transformation then

$g(x) = (x_1 + \mu, x_2 + \mu, \dots., x_n + \mu)$ , $\quad -\infty < \mu < \infty$, then

$$T(x) = T\big(g(x)\big) = (x_n - x_1, \dots, x_n - x_{n-1}).$$

is called as maximal location invariant estimator.

### *2.2.2 Scale Invariant*

Let $x = (x_1, x_2, \dots, x_n)$, be the random sample from scale family with scale parameter $\sigma$ and $G$ be the group transformation then
$g(x) = (\sigma x_1, \sigma x_2, \dots., \sigma x_n)$ , $\quad -\infty < \mu < \infty$, then

$$T(x) = T\big(g(x)\big) = (\frac{x_n}{x_1}, \frac{x_1}{x_2}, \dots, \frac{x_{n-1}}{x_n}).$$

$T(x)$ is maximal scale invariant estimator.

### *2.2.3 Location-Scale Invariant*

Let $x = (x_1, x_2, \dots, x_n)$, be the random sample from location-scale family with location parameter $\mu$ and scale parameter $\sigma$. Let $G$ be the group transformation then
$g(x) = (\sigma(x_1 + \mu), \sigma(x_2 + \mu), \dots., \sigma(x_n + \mu))$ , $\quad -\infty < \mu < \infty$, then

$$T(x) = T\big(g(x)\big) = (\frac{x_n - x_{n-1}}{x_2 - x_1}, \frac{x_{n-1} - x_{n-2}}{x_3 - x_2}, \dots, \frac{x_2 - x_1}{x_n - x_{n-1}}, \frac{x_1 - x_n}{x_n - x_1}).$$

$T(x)$ is maximal location-scale invariant estimator.

The next section reviews the literature related to the treatment for nuisance parameters.

## 3. LITERATURE REVIEW

There have been numerous articles addressing a systematic study of a variety of methods for eliminating nuisance parameters.

### 3.1. Likelihood Based Approach

A pseudo-likelihood or profile likelihood is obtained by replacing the nuisance parameters with their maximum likelihood estimators obtained by keeping the parameters of interest fixed. After fixing the interest parameters, the MLEs of nuisance parameters are expressed as functions of interest parameters and after replacing the nuisance parameters by these functions, the likelihood gets translated to a function of only interest parameters. This likelihood behaves similar to the classical likelihood. For the critical review and various aspects of pseudo or profile likelihood, we refer to Kalbfleish and Sprott (1989)[21], Gong and Samaniego (1981)[22], Fraser and Reid (1989)[23], Barndorff-Nielsen (1985)[24], Barndorff-Nielsen (1991)[25], Barndorff-Nielsen (1994)[26] and Severini (1998)[27].

Integrated likelihood approach is another way to eliminate nuisance parameters, For notable analytical results in this context we refer to Berger and Wolpert (1988), Berger et al. (1999), Severini (2000), and Severini (2010), among others. Notable novel recent inferential procedures based on integrated likelihood have been developed by SenGupta and Kulkarni (2018), Kulkarni and SenGupta (2021), Patil and Kulkarni (2022), and Kulkarni and Patil (2021) under directional and linear data[23-27].

### 3.2. Invariance Principle Approach:

Nuisance parameters free inference can also be based on an ancillary statistic, invariant or weighted average power criterion, and conditional probability as reported in Linnik and Technica (1968), Cox and Hinkley (1974), Engelhardt and Bain (1977), Andrews and Ploberger (1994), and Hansen (1996)[28].

Invariance principle can be coupled with appropriate data transformation to yield nuisance parameters free transformed likelihood that is purely function of the parameters of interest and the observed sample only. Zaigraev and Podraza-Karakulska (2008) addressed the maximal scale invariant estimation procedure for the shape parameter of gamma distribution. Kulkarni and Patil (2018a) derived maximal scale invariant inference for the shape parameter under shape-scale family of distributions[29].

Tsui and Weerahandi (1989) developed the concept of generalized test variable (GTV) and generalized p-value (GPV) for significance testing based on a suitable generalized extreme region where the p-value is independent of the nuisance parameters[30]. Exact statistical inference based on GTV, GPV, and generalized confidence interval (GCI) can be found in Weerahandi (1995). Hannig et al. (2006) identified an important subclass of generalized pivotal quantities (GPQ) which have asymptomatically correct frequentist coverage. Nkurnziza and Chen (2011) provide a systematic approach to construct GPQ, GCI, and GPV for a location-scale family of distributions[30].

The present work reviews univariate, two-sample, and multi-sample improved procedures that efficiently handle the nuisance parameters and the recommended procedures are given in the next section.

## 4. IMPROVED INFERENTIAL PROCEDURES

Kulkarni and Patil (2018a)[31] introduced the maximal scale-invariant estimation procedure for the shape parameter of the shape-scale family of distributions. The method for obtaining nuisance parameters-free likelihood for the shape parameter based on maximal scale-invariant transformation for eliminating the nuisance scale parameter is explained. The resulting likelihoods are functions of only the shape parameter of interest. The results are illustrated for popular shape-scale distributions, namely the Weibull, the Gamma and the Generalized exponential (GE) distribution under complete and type-II censored samples. The proposed maximal scale-invariant likelihood estimator (MSILE) for the shape parameter of interest, being based on a proper likelihood function enjoys all asymptotic properties under regular conditions[31].

A simulation study for the Weibull and Gamma distributions revealed an almost exact relationship between the bias of the MSILE and the maximum likelihood estimator (MLE). An improved, almost unbiased estimator (AUE) is proposed by exploiting this linearity. The extent of reduction in bias and mean square error (MSE) of the MLE, MSILE and AUE reveals the superiority of MSILE over MLE, and the superiority of AUE over MSILE and MLE for Weibull and Gamma distribution[32]. One-sample test and $100(1 - \alpha)\%$ confidence interval for the shape parameter is developed, and performance is assessed with respect to the observed size of relevant test procedures, and coverage probability and average width of the associated confidence interval. Furthermore, the MLE of the scale parameter being a function of the shape parameter, is obtained by replacing the shape parameter with its MSILE. The performance of the resulting estimator was observed to be superior than its regular MLE[33].

The interval estimation for the stress-strength reliability (R) under the exponentiated-scale family of distributions is developed in the Patil and Kulkarni (2018)[34]. The exponentiated-scale family was introduced by Marshall and Olkin (2007), which is also known as resilience or frailty parameter family. The distributional form of resilience family is:

$$G\left(\frac{x}{\lambda}, \alpha\right) = F^{\alpha}\left(\frac{x}{\lambda}\right),$$

$\alpha$ being a resilience parameter, while the distributional form of frailty family is:

$$\bar{G}\left(\frac{x}{\lambda}, \alpha\right) = \bar{F}^{\alpha}\left(\frac{x}{\lambda}\right),$$

$\alpha$ being a frailty parameter, $\lambda$ the scale parameter, and $F\,(.)$ is a known distribution function while $\bar{F}\,(.)$ is the corresponding survival function.

The stress–strength reliability $R = P\,(X_1 < X_2)$ where $X_1$ and $X_2$ represent the stress applied and strength of an equipment, respectively, plays a crucial role in setting warranty periods while launching new brands of a product, among other applications. Patil and Kulkarni (2018) address the issue of estimating $R$ when $X_1$ and $X_2$ belong to the exponentiated scale family, which includes the popular Exponentiated-exponential distribution (EED) that has proven to be an excellent model for lifetime distributions. The cases of known/unknown and equal/unequal scale parameters are handled separately. For equal scale parameters of $X_1$ and $X_2$ the expression for $R$ turns out to be purely function of the shape parameters. When the scale parameters are unequal the reliability $R$ turns out to be a function of the underlying shape parameter and ratio of the scale parameters. For known scale parameter, a generalized pivot quantity for the shape parameter and $R$ are developed. The interval estimates of $R$ based on the proposed generalized pivot quantity exhibited uniformly best performance. For an unknown scale parameter, a maximum scale invariant likelihood estimator of the shape and an allied estimator of the scale are introduced. An extensive simulation-based comparison is performed among following five methods:

GPQ: Generalized pivotal quantity.

PBMSILE: A parametric bootstrap technique employed on MSILE.

PBMLE: A Parametric bootstrap technique employed on MLE.

NPBMSILE: A nonparametric bootstrap technique employed on MSILE.

NPBMLE: A nonparametric bootstrap technique employed on MLE.

The parametric bootstrap interval estimates of $R$ based on the proposed maximum scale invariant likelihood estimator of the shape parameter exhibited best performance among others. An application in setting warranty periods is illustrated based on two real data sets[35].

Micro-array experiments are important fields in molecular biology where zero values mixed with a continuous outcome are frequently encountered leading to a mixed distribution with a clump at zero. Comparison of two mixed populations, for example of a control and a treated group; of two groups with different types of cancer, to name a few, are often encountered in these contexts. Fairly skewed distribution of the continuous part coupled with small sample sizes are issues of main concern to be attended for the quality of inference in such situations. However, popularly used non-parametric methods rely on asymptotic distribution of the underlying test statistics which are valid only under large sample sizes. Kulkarni and Patil (2018b) address the aforementioned issues via a newly proposed exact test for location-scale family distributions and GPQ based parametric test procedures for non-location-scale distributions. The proposed test procedure can be used under a best fitted continuous distribution. It consists of k+1 parts, where k is the number of parameters for a specific best fitting parametric model used for the continuous component. More specifically, the first part tests the equality of the proportions of zeros while the remaining k parts test the equality of the k corresponding individual parameters in the two populations under consideration. Note that the combined test is equivalent to testing equality of the two entire mixed populations under consideration. The k+1 parts and their combination produce an overall p-value for testing the combined hypothesis of equality of the two distributions. In order to account for the dependency among simultaneous testing of a large number of tests, we calibrate the observed p-values using the Benjamini–Hochberg (1995) procedure[36].

A simulation study is carried out for validation and performance evaluation of the proposed exact test for location-scale or log-location-scale family of distributions and GPQ based test for non-location-scale distributions. The proposed test is compared with the popular two-part (TP) test based on the type-I error and power of the tests. The TP test consists of two parts one is of testing equality of proportions of zeros and other non-parametric test comparing two continuous data sets. Different tests are used to compare the continuous part, namely Kolmogorov- Smirnov, t-test, Wilcoxon rank sum test, Ansari Bradley test, Sigel-Tukey test[37].

Simulation based assessment of the proposed exact test based on invariance principle for location-scale family distributions and GPQ based parametric test procedures for non-location-scale distributions showed their superior performance with respect to size and power in comparison to the above popular two-part tests, more prominently for small sample sizes[38].

A number of distributions including the Exponential, Extreme value, Normal, Double exponential, Inverse Gaussian, Weibull, Pareto, Log-Normal and Gamma distributions have been handled to illustrate the above testing procedure for microarray data. We could identify 1555 differentially expressed genes[39].

Future scope on RNA sequence count data analysis through the GPQ and GTV for Poison and Negative binomial parameters is discussed, and a generalized test procedure is suggested for two discrete populations in similar lines.

Patil and Kulkarni (2022) developed a unified approach for testing homogeneity of variances among k ($k > 2$) independent location-scale populations. The proposed test is based on a generalized test variable. The GPV for testing homogeneity of variances is obtained by constructing GPQs for the k distinct scale parameters of the k populations. The performance of the proposed test is assessed through an extensive simulation study on popular location-scale families in comparison to the existing tests. The proposed test is uniformly superior over existing popularly used parametric and non-parametric tests in terms of type-I errors and power function. A systematic study to assess the impact of the extent of kurtosis and skewness is made through simulation studies under the Generalized Normal and Skew Normal distributions respectively[40-41].

A uniformly implementable small sample integrated likelihood ratio test for one way and two-way ANOVA under heteroscedasticity and normality is developed by Patil and Kulkarni (2021) which has an asymptotic chi-square distribution up to second order accuracy. Simple ad hoc corrective adjustments recommended for improving the small sample distributional performance make the test usable even for very small group sizes. Empirical assessment of the test reveals that the test exhibits uniformly well-concentrated sizes at the desired level and the maximal power, particularly under very small size groups. In similar lines, Patil and Kulkarni (2022) develop a test for analysis of medians for Birnbaum–Saunders distributed response to assess the impact of two interacting factors on the median, where no any test available in the literature.

Ma et. al. (2022) studied the statistical inference on the location parameter vector in the multivariate skew-normal model with unknown scale parameter and known shape parameter. Based on the distribution of the generalized Hotelling's $T^2$ statistic, confidence regions and hypothesis tests on the location parameter $\mu$ are obtained[42].

## 5. RECOMMENDATIONS

The GPQ or Fiducial approach-based procedures or invariance-based procedures are recommended as the best alternative to classical or popularly used inferential procedures in the presence of nuisance parameters and often work well even under small sample sizes. A maximal scale invariant inference for shape and allied inference on scale parameter is a substitute for classical maximum likelihood point and interval estimation as well as testing problem under shape-scale and exponentiated-scale family of distributions. Generalized variable approach and a maximal scale invariant transformation-based inference is recommended for the stress-strength reliability under exponentiated-scale family of distributions. Exact test based on fiducial inference is recommended for Comparison of two continuous populations mixed with point mass at zero and to test the homogeneity of variances among several independent location-scale populations. When GPQ/invariance principle-based procedures are not available, among the likelihood-based procedures, the integrated likelihood principle works the best.

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